

Applications of Mathematical techniques to The FitzHugh-Nagumo Equation

Problem Description

I would be applying mathematical techniques in analyzing the behaviour of solutions of the FitzHugh Nagumo. The FitzHugh Nagumo are a pair of first order differential equations which model the potential in a nerve. The model is used to conceptually represent the “mathematical properties of excitation and propagation from the electrochemical properties of sodium and potassium ion flow in the body.” (Eugene M, n.d). For this analysis, we’ll be working with the variant of the FitzHugh-Nagumo below; as shown in the Holmes textbook.

$$v' = c \left(v - \frac{1}{3}v^3 + w \right)$$
$$w' = -\frac{1}{c}(v - a - bw)$$

Where v represents the action potential in the nerve, w represents a recovery variable, and a, b and c are constants. Since both equations show the first order differential of the functions for v and w with respect to time, integrating both functions with respect to time should give the functions for v and w at anytime t . However, we would be applying other mathematical techniques in understanding the behaviour of this system and its solutions.

Applied Mathematical Techniques

I would approach this set of first order differential equations using 3 different mathematical techniques. First, I transform the given set of equations into an initial value problem by stating values for the constants a, b and c , and initial values for v and w . Then, I apply methods of solving initial value problems on the newly generated IVP, including Euler’s method, trapezoidal method, RK4 and Velocity Verlet methods. These methods can then be compared, and the most appropriate method would be used in the next analysis.

For the next step; the solution from the most appropriate method would be used in generating v s and w s within a time range (0 to 20 minutes). An interpolation technique of choice could then be applied on the generated data. This interpolation technique could be a cubic spline, or lagrange polynomial function, or even both. The effectiveness and behavior of the interpolation technique would be observed and reported.

Finally, an analysis of the data through the lens of principal component analysis would be done, in order to gain more insight on the connection between the v and w variables - if possible. This would be the final aspect of the analysis of the FitzHugh Nagumo equations.

Experiments which compare numerical techniques, together with analysis of results

Experiment 1: Solving the IVP using mathematical techniques

In this experiment, we would use the values given in the Holmes textbook, ie set $a = b = 0.2$, $c = 3$ and initial values $w(0) = 1$ and $v(0) = -1$ to create an IVP. We then proceed to briefly describe 5 techniques which could be applied towards solving the IVP, and then finally code up 4 of those 5 techniques.

Case 1: Euler's method

The Euler's method is an explicit, conditionally A-stable method of solving IVPs. It is a recursive method, and involves starting from $j = 0$ and finding the solution of the IVP at $j=1$ to $j=N$. It has a truncation term of $O(k)$. Applying it on the above problem, we have solutions to the IVP taking the form

$$v_{j+1} = v_j + kc(v_j - \frac{v_j^3}{3} + w_j)$$

$$w_{j+1} = w_j - \frac{k}{c}(v_j - a - bw_j)$$

Case 2: Leapfrog method

The Leapfrog method is an explicit method of solving IVPs. It is called a leapfrog method because it incorporates information from the previous two time steps in evaluating the value of

the function at the present time. It has a truncation term of $O(k^2)$, and is an unstable method of solving IVPs. Applying it to the above problem, we have the form of the solutions below;

$$v_{j+1} = v_{j-1} + 2kc(v_j - \frac{v_j^3}{3} + w_j)$$

$$w_{j+1} = w_{j-1} - \frac{2k}{c} (v_j - a - bw_j)$$

Case 3: Trapezoidal Method

The Trapezoidal method is an implicit method of solving IVPs. It is also A-stable, and has a truncation term of $O(k^2)$. It is an iterative method which is named the trapezoidal method because it was generated using the trapezoidal rule finite difference approximation. Applying it to the FitzHugh Nagumo equation, we get the form below;

$$v_{j+1} = v_j + \frac{kc}{2} ((v_j - \frac{v_j^3}{3} + w_j) + (v_{j+1} - \frac{v_{j+1}^3}{3} + w_{j+1}))$$

$$w_{j+1} = w_j - \frac{k}{2c} ((v_j - a - bw_j) + ((v_{j+1} - a - bw_{j+1})))$$

Case 4: Velocity Verlet Method

The Velocity verlet Method is one of the methods of solving IVPs, which is explicit and symplectic. In a bid to reduce the implicitness of the Trapezoidal method, and thus reduce runtime, it is a combined form of the Euler's method and the trapezoidal method. It has an order of $O(k^2)$ and is a stable method of solving IVPs. When applied on this problem, we get the solution;

$$v_{j+1} = v_j + kw_j + \frac{k^2c}{2} (v_j - \frac{v_j^3}{3} + w_j)$$

$$w_{j+1} = w_j - \frac{k}{2c} ((v_j - a - bw_j) + ((v_{j+1} - a - bw_{j+1})))$$

Case 5: The RK4 Method

The RK4 method is one of the RK methods of solving IVPs, with a truncation error of $O(k^4)$. It is a conditionally A-stable method, and also an explicit method. It is based on the trapezoidal formulation, but solves the problem of implicitness by replacing implicit terms with approximations for those terms. Applying it to the problem at hand, we get the solution below.

$$v_{j+1} = v_j + \frac{1}{6}(k1 + 2k2 + 2k3 + k4)$$

where

$$k_1 = kf(t_j, v_j),$$

$$k_2 = kf(t_j + \frac{k}{2}, v_j + \frac{1}{2}k_1),$$

$$k_3 = kf(t_j + \frac{k}{2}, v_j + \frac{1}{2}k_2),$$

$$k_4 = kf(t_{j+1}, v_j + k_3)$$

$$f(t_j, v_j) = c(v - \frac{1}{3}v^3 + w)$$

and

$$w_{j+1} = w_j + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

where

$$k_1 = kF(t_j, w_j),$$

$$k_2 = kF(t_j + \frac{k}{2}, w_j + \frac{1}{2}k_1),$$

$$k_3 = kF(t_j + \frac{k}{2}, w_j + \frac{1}{2}k_2),$$

$$k_4 = kF(t_{j+1}, w_j + k_3).$$

$$F(t_j, w_j) = -\frac{1}{c}(v - a - bw)$$

Comparison of the 4 different techniques used in solving the problem

The MATLAB software was used in programming solutions to the above IVP problem. 4 of the 5 techniques which were transformed into matlab code, and used in solving the IVPS are the Euler's method, Trapezoidal, Velocity verlet and RK4 method. In this part of the analysis, the table and graph below compare the 4 different methods according to various metrics.

Metric	Euler method	Trapezoidal Method	Velocity verlet Method	RK4 Method
Implicit vs Explicit	Explicit	Implicit	Explicit	Explicit

Stability	Conditionally A-stable	A-stable	A-Stable	Conditionally A-stable
Structure	Basic recursive	Recursive, uses trapezoidal rule	Combines Euler and Trapezoidal methods to reduce implicitness	Uses Trapezoidal method, but eliminates implicitness using approximations
Runtime for t = 20	0.0249419	0.05319	0.0228801	0.0500522
Graph similarity to other methods	Has the same shape as other graphs and has closer values to the Trapezoidal graph. Has the smallest extrema than other graphs	Has the same shape as the other graphs. Starts off close to the euler graph, but shifts out of phase with the other graphs eventually	Begins within the range as other graphs, but slowly gets out of phase with graphs from different solutions	Begins out of phase than the other methods, and remains out of phase. Has larger extrema values than other graphs
Truncation Error	$O(k)$	$O(k^2)$	$O(k^2)$	$O(k^4)$

Fig 1: Table comparing the different techniques of solving IVPs when applied to the FitzHugh equation

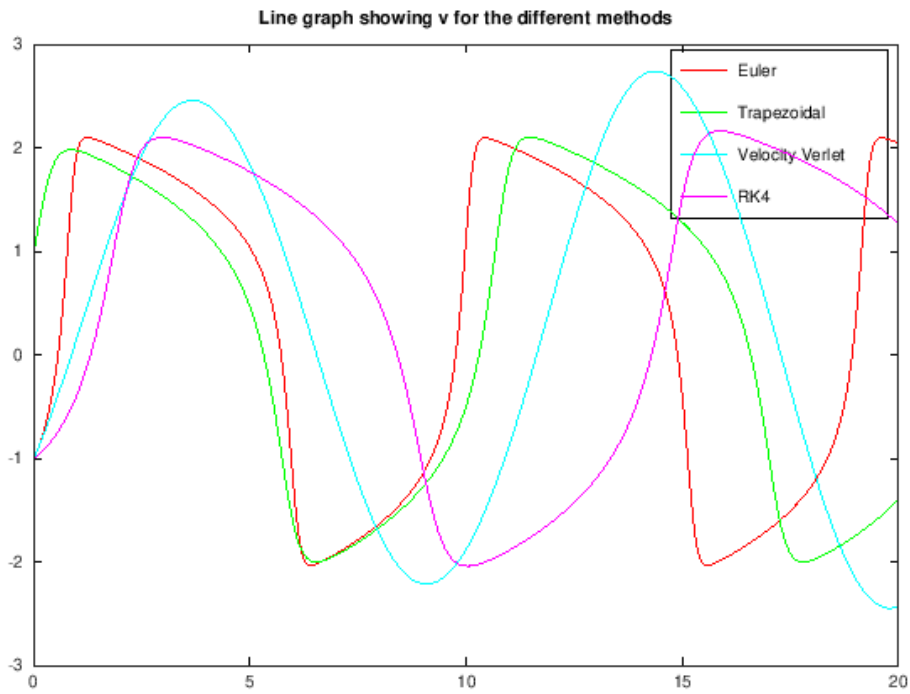


Fig 2: Graph showing the value of v for the different methods of solving IVPs

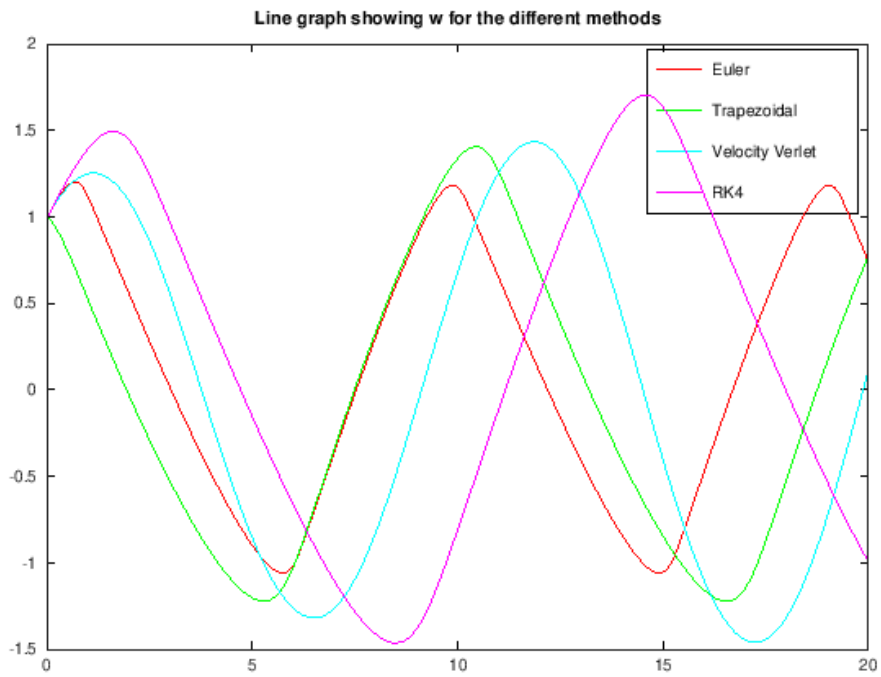


Fig 2: Graph showing the value of w for the different methods of solving IVPs

Experiment 2:

For the sake of the next set of experiments, I chose to move on with results from the RK4 method. This is mostly because the RK4 method has the lowest order of the 4 methods, with a truncation error of $O(k^4)$. When combined with floating point errors gotten during the process of calculations, this would mean that the RK4 method has the smallest overall error when compared with the other 3 techniques. Recall that floating point errors are errors which arise as a result of round off during the process of conversion of numbers to their floating point representations (forms which the computer can work with).

The focus of this experiment would be to apply an interpolation technique to the data gotten from the RK4 method, and observing the behaviour of the results. For clarity on the process, a scatter plot of the data gotten from the RK4 method is plotted below;

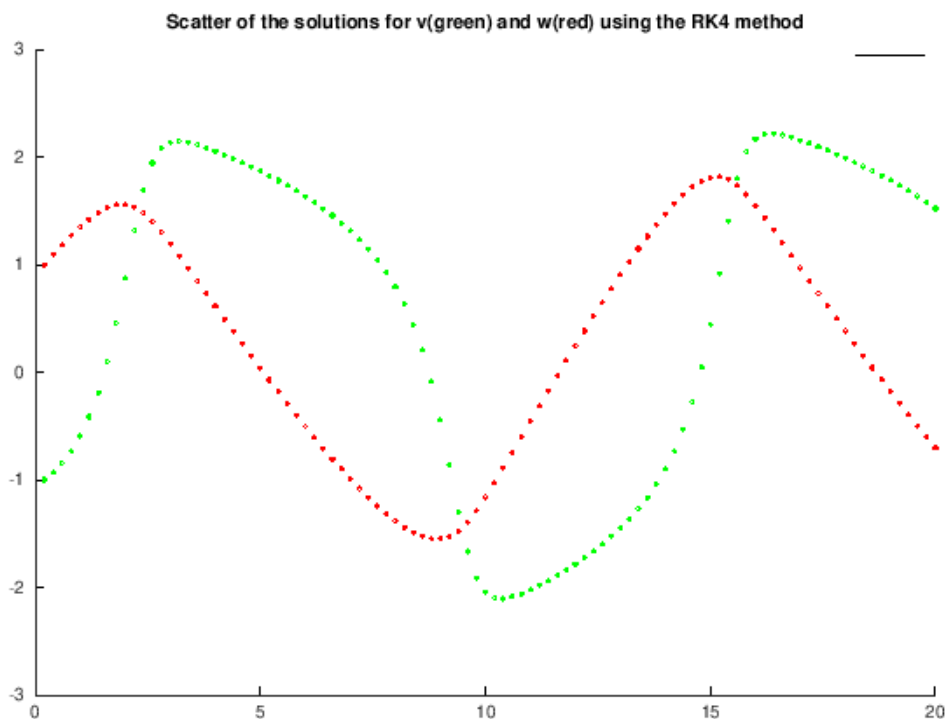


Fig 3: Scatter plot showing the values of v and w at time t using the RK4 method.

The cubic spline interpolation method is used for the interpolation. Along with the fact that this method is an inbuilt matlab function, it is easier to compute when compared to methods like the Lagrange polynomial method(which can have problems with larger data sets) or the Vandermonde matrix method for which it would be difficult to solve the $Va = y$ equation involving the Vandermonde matrix, especially for large values of the input. We must note too that the variant of the cubic spline interpolation method which we would be using is the not-a-knot spline, as specified by matlab. Plotting the results, we see that the interpolation does a great job.

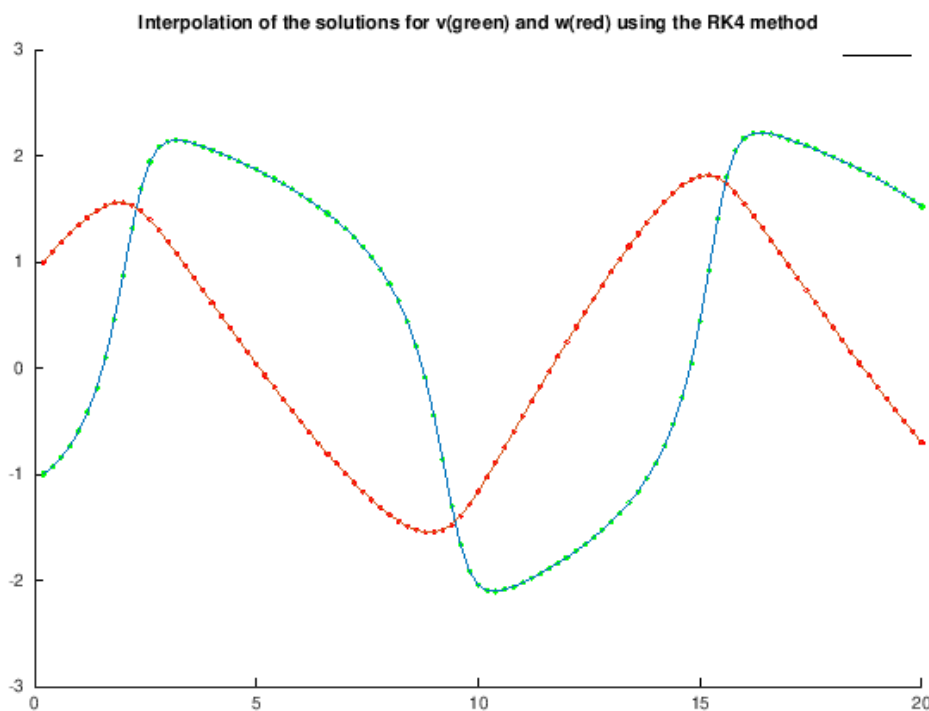


Fig 4: Graph showing the use of piecewise interpolation on the v and w data from the RK4 method.

Applying the error function given in the Holmes textbook, we get an error value of 0. This means that the cubic spline interpolation, in fact, does an excellent job in interpolating the data.

$$E(c, b) = \sum_{i=1}^n \{[v(t_i) - v_i]^2 + [w(t_i) - w_i]^2\}. \quad (9.52)$$

Based on these results, we can be satisfied that we have been able to generate data for the values of the action potential of a nerve, and the recovery variable at any time t using the RK4 method, and we've also been able to find a function that interpolates these generated data using the inbuilt not-a-knot cubic spline method. Finally, we perform our last experiment on this data.

Experiment 3: Use of Principal Component Analysis on the Data

In this analysis, we would like to know if there's a connection between the action potential of a nerve (v) and the recovery variable (w). Although there is an obvious connection between the two as seen in the IVP problem, this connection isn't clear enough, and we seek a much clearer connection through the lens of a principal component analysis. To proceed with this, first we try looking at what the scatter plot of the v variable against the w variable looks like. This result would be relevant in knowing whether a PCA analysis would be feasible on the given data or not.

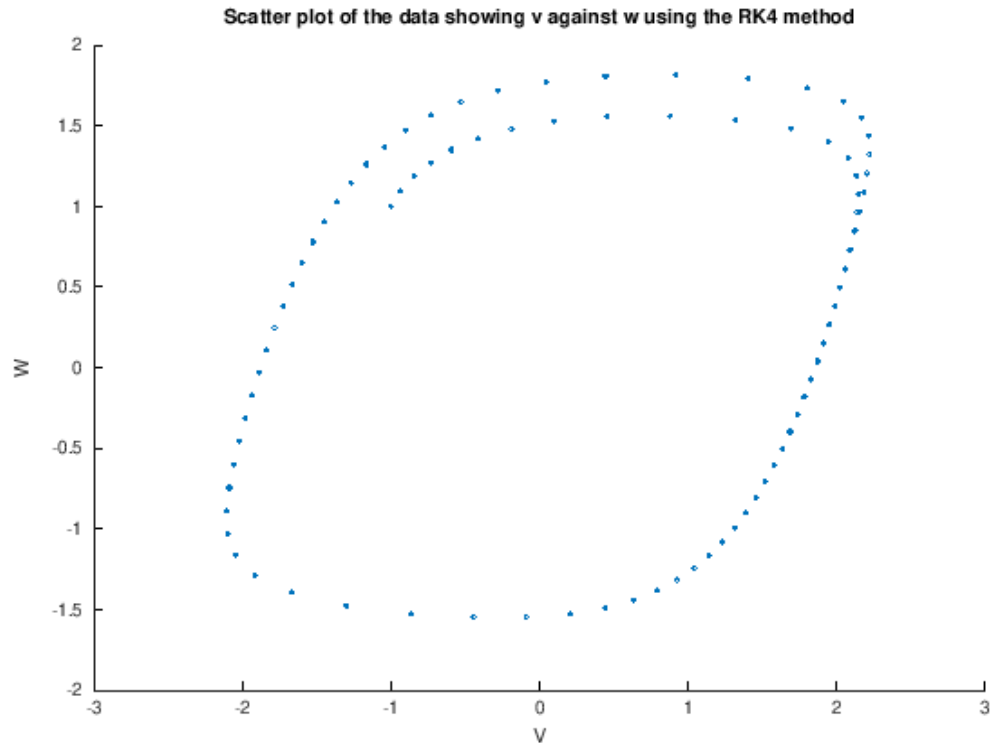


Fig 5: Scatter plot showing the data from the RK4 method, with v on the x axis and W on the y axis

From an observation of the data, we can conclude that this scatter plot has less of an ellipsoidal shape, and more of a circular or spherical shape. This would mean that although the PCA might work in this case, it wouldn't produce an effective solution to the problem. This is because there could be more than one line of best fit. The lack of uniqueness of this line of best fit would also mean that the connection between the v and w variable wouldn't be unique using the PCA. To solve this problem, as suggested by the Holmes textbook, one would need to use a two or more dimensional analysis for the PCA, instead of a one dimensional analysis. However, this still can't be applied to this problem, because there are only 3 variables in the FitzHugh Nagumo equation, and we have already performed a critical analysis of the relationship between time and the other 2 variables in previous parts of this paper.

In conclusion, although a principal component analysis would be quite an interesting

analysis to make, in order to give a clearer understanding of what the connection between v and w is, the available data doesn't make this analysis easily feasible.

Summary

In this paper, we have performed a critical analysis of the FitzHugh Nagumo equation. First, we begin with an IVP representation of the equations. Then we proceed to solve this IVP using different mathematical techniques. Next, we compare the various solutions obtained, and consequently use the RK4 method to generate data for later parts of the analysis because of its low truncation error. We then use a not-a-knot cubic spline interpolation method to fit the data generated from the RK4 method. Finally, we attempt to apply a principal component analysis on the given data, to find the connection between the action potential of a nerve, and the recovery variable. However, we end up discovering that this isn't feasible, and wouldn't give us the required results that we seek.

In general, this was an interesting assignment, and has given more insight into what the action potential in a nerve is at any point in time. For example, we have been able to observe that the solution of the FitzHugh Nagumo equation is a relaxation oscillator, because of the nonsinusoidal repetitive form of the graph.

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